Announcements

1) Math Advising Session
11:30 - 1:30 CB 2047
(Math Library) - There
will be pizza!

2) Math Awards Ceremony
2:00, CB 2046

3) The Final likely
to be moved

Tangent Spaces and Forms

Let $p \in \mathbb{R}^n$. We define the tangent space to p as the vector space of all pairs (p, v) with $v \in \mathbb{R}^n$

- $c(p, v) = (p, cv)$, $c \in \mathbb{R}$
- $(p, v_1) + (p, v_2) = (p, v_1 + v_2)$ for $v_1, v_2 \in \mathbb{R}^n$

Definition: $(\mathbb{R}^n_p)$

$$\mathbb{R}^n_p = \{(p,v) \mid p, v \in \mathbb{R}^n, p \text{ fixed}\}$$

with its vector space structure.

Spivak sometimes writes

$$v_p = (p, v).$$

Definition: (vector field)

A **vector field** is a function F from $\mathbb{R}^n$ such that $F(p) \in \mathbb{R}^n_p$ $\forall$ $p \in \mathbb{R}^n$. We can write

$$F(p) = F_1(p)(p, e_1) + \cdots + F_n(p)(p, e_n)$$

$\forall$ $p \in \mathbb{R}^n$ and for some function $F_i : \mathbb{R}^n \to \mathbb{R}$ $\forall$ $1 \leq i \leq n$.

If each F_i is differentiable,
we say F is differentiable.

Example 1: (gradient field)

Let $f: \mathbb{R}^n \to \mathbb{R}$.

Then $Df = \nabla f$

$$= \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right\rangle$$

We obtain a vector field by

$$F(p) = (p, \nabla f(p))$$

$$f(x,y) = x\sin(xy)$$

$$\frac{\partial f}{\partial x} = \sin(xy) + xy\cos(xy)$$

$$\frac{\partial f}{\partial y} = x^2\cos(xy)$$

If $p = (p_1, p_2)$,

$F(p)$

$$= \left((p_1, p_2), \left(\sin(p_1p_2) + p_1p_2\cos(p_1p_2),\ p_1^2\cos(p_1p_2)\right)\right)$$

Definition: (divergence and curl)

Let F be a vector field on $\mathbb{R}^n$. We define $\mathrm{div}(F)$, the divergence of F, as the scalar quantity

$$\boxed{\mathrm{div}(F) = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \cdots + \frac{\partial F_n}{\partial x_n}}$$

if $F_1, F_2, \ldots, F_n$ are the components of F and F is differentiable.

If $n=3$, we define the curl of F to be

$\nabla \times F$ where

"∇" is the formal rector

$$\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle.$$

Definition: (k-form, general, differentiable)

A general k-form on $\mathbb{R}^n$ is a map from $\mathbb{R}^n$ into $\Lambda^k(\mathbb{R}^n_p)$. So if ω is such a k-form,

$$\boxed{\omega(p) = \sum_{i_1 < i_2 < \cdots < i_k} \omega_{i_1, \ldots, i_k}(p)\, \varphi_{i_1,p} \wedge \cdots \wedge \varphi_{i_k,p}}$$

where $\varphi_{i,p}$ are the functionals φ_i represented on $\mathbb{R}^n_p$:

$$\boxed{\varphi_{i,p}(p,v) = \varphi_i(v).}$$

We say ω is differentiable if $\omega_{i_1, \ldots, i_k}$ is a C^∞ (infinitely continuously differentiable) function from $\mathbb{R}^n$ to $\mathbb{R}$ $\forall$ $i_1 < i_2 < \cdots < i_k$.

Note: (0 form and 1 forms)

A zero form is just a function from $\mathbb{R}^n$ to $\mathbb{R}$.

A 1-form is given by the derivative of a (differentiable) 0-form. We want to extend this to k-forms.

Notation: (dx_i)

dx_i is the form given by

$$dx_i(p) = \varphi_{i,p}$$

Definition: (d)

Given a differential k-form

$$\omega = \sum_{i_1 < i_2 < \cdots < i_k} \omega_{i_1, \ldots, i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

we define

$$\boxed{d\omega(p) = \sum_{i_1 < i_2 < \cdots < i_k} d\omega_{i_1, \ldots, i_k}(p) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}}$$

where $d\omega_{i_1, \ldots, i_k}(p)(p, v)$
$= (D\omega_{i_1, \ldots, i_k}(p))(v)$

Definition: (f^*)

$f : \mathbb{R}^n \to \mathbb{R}^m$, C^∞

$Df(p) : \mathbb{R}^n \to \mathbb{R}^m \quad \forall\, p \in \mathbb{R}^n$

Define

$f_* : \mathbb{R}^n_p \to \mathbb{R}^m_{f(p)}$

$$f_*(p, v) = \big(f(p), Df(p)v\big)$$

We define

$$f^*: \Lambda^k(\mathbb{R}^m_{f(p)}) \to \Lambda^k(\mathbb{R}^n_p)$$

by

$$f^*(\omega)(p,v)$$
$$= \omega(f_*(p,v))$$
$$= (\omega \circ f_*)(p,v)$$

Finally, define

$f^*\omega$ to be

$$(f^*\omega)(p) = f^*(\omega(f(p)))$$

$f^*\omega$ is a k form on $\mathbb{R}^n$

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for ω a k form on $\mathbb{R}^m$

Integration

Let $I_n = \prod_{i=1}^{n} [0,1]$,

the standard n-cube

If ω is a n-form on $\mathbb{R}^n$,

$\omega(p) = f(p)\, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$

$\in$

$\Lambda^n(\mathbb{R}^n_p)$

We define

$$\boxed{\int_{I^n} \omega = \int_{I^n} f}$$

Definition: (singular n-cube)

A map $c : I_n \to \mathbb{R}^m$

is called a **singular n-cube**

if c is continuous.

Definition: (n-chain)

An n-chain is a "formal sum" of singular n-cubes.

If $c_1, \dots, c_m$ are singular n-cubes, any singular n-chain is of the form

$$a_1 c_1 + a_2 c_2 + \dots + a_m c_m$$

for some $a_1, \dots, a_m \in \mathbb{R}$.

The addition is just a way to keep track of how you integrate over n-chains.

Definition: If c is a singular n-cube and ω is an n-form, we define

$$\int_c \omega = \int_{I^n} c^* \omega$$

Definition: (boundary)

If I_n denotes the n-cube, define the "face maps"

$I_{(i,0)}$ and $I_{(i,1)}$ for $1 \leq i \leq n$ as singular $(n-1)$ cubes,

$$I_{(i,0)}, I_{(i,1)} : I_{n-1} \to \mathbb{R}^n$$

$$I_{(i,0)}(x_1, --, x_{n-1})$$

$$= (x_1, x_2, \cdots, x_{i-1}, 0, x_i, --, x_{n-1})$$

$$I_{(i,1)}(x_1, ---, x_{n-1})$$

$$= (x_1, x_2, -, x_{i-1}, 1, x_i, -, x_{n-1})$$

We then define the **boundary** of I^n

as

$$\partial I^n = \sum_{i=1}^{n} \left((-1)^i I_{(i,0)} + (-1)^{i+1} I_{(i,1)} \right)$$

If c is a singular n-cube, we define

$$\partial c = \sum_{i=1}^{n} (-1)^i c \circ I_{(i,0)} + (-1)^{i+1} c \circ I_{(i,1)}.$$

Finally, given an n-chain $c = \sum_{i=1}^{m} a_i c_i$, define

$$\partial c = \sum_{i=1}^{m} \alpha_i \partial c_i$$

Theorem: (Stokes, general)

Let ω be an $(n-1)$ form on $\mathbb{R}^m$.

Let c be an n-chain.

Then

$$\boxed{\int_{\partial c} \omega = \int_c d\omega}$$

Example 2: (fundamental theorem of calculus, 1-D)

$$I_1 = [0,1]$$

$$\partial I_1 = \{0\} \cup \{1\}$$

If ω is a 0-form,

ω is some C^∞ function $f: \mathbb{R} \to \mathbb{R}$ and

$d\omega$ is its derivative.

Stokes' theorem says

$$\int_0^1 df = \int_{\partial([0,1])} f$$

$$= f(1) - f(0)$$

by definition

Example 3: (Green's Theorem)

ω is a 1 form on $\mathbb{R}^2$,

so $\omega = P\,dx + Q\,dy$

for some differentiable $P, Q : \mathbb{R}^2 \to \mathbb{R}$

$$d\omega$$

$$= \left(\frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy\right) \wedge dx$$

$$+ \left(\frac{\partial Q}{\partial x} dx + \frac{\partial P}{\partial y} dy\right) \wedge dy$$

$$= \frac{\partial P}{\partial x}(\underbrace{dx \wedge dx}_{=0}) + \frac{\partial P}{\partial y}(\underbrace{dy \wedge dx}_{=-dx \wedge dy})$$

$$+ \frac{\partial Q}{\partial x}(dx \wedge dy) + \frac{\partial Q}{\partial y}(\underbrace{dy \wedge dy}_{=0})$$

$$= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy$$

which complies with what you should get in Green's theorem!

Proof of Stokes, $n=2$

We reduce to $C = I^2 = [0,1]\times[0,1]$

Then

$$\int_{[0,1]\times[0,1]} d\omega = \int_{[0,1]\times[0,1]} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy$$

$$= \int_0^1\int_0^1 \frac{\partial Q}{\partial x}\, dx\,dy - \int_0^1\int_0^1 \frac{\partial P}{\partial y}\, dy\,dx$$

by Fubini's Theorem,

which then equals

$$\int_0^1 (Q(1,y)-Q(0,y))\,dy$$

$$-\left(\int_0^1 (P(x,1)-P(x,0))\,dx\right)$$

$$= \int_0^1 Q(1,y)\,dy - \int_0^1 Q(0,y)\,dy$$

$$- \int_0^1 P(x,1)\,dx + \int_0^1 P(x,0)\,dx$$

That was the easy part!

Now from the definition,

$$\partial c = -I^2_{(1,0)} + I^2_{(1,1)} + I^2_{(2,0)} - I^2_{(2,1)}$$

So then

$$\int_{\partial c} \omega$$

$$= -\int_{I^2_{(1,0)}} \omega + \int_{I^2_{(1,1)}} \omega + \int_{I^2_{(2,0)}} \omega - \int_{I^2_{(2,1)}} \omega$$

4 integrals - promising!

To be perverse, let's focus on the 3rd integral.

By definition

$$I^2_{(2,0)} : [0,1] \to I^2, \text{ so}$$

$$\int_{I^2_{(2,0)}} \omega = \int_0^1 {I^2_{2,0}}^* \omega$$

Spivak proves many preliminary results to make this computation easier. We'll just bare-hands it.

We know $I_{(2,0)}^{2\,*}\omega$ is

a 1 form on $\mathbb{R}$ since ω is a 1 form on $\mathbb{R}^2$. Therefore,

$\exists$ a function $f: \mathbb{R} \to \mathbb{R}$ with

$$\left(I_{(2,0)}^{2\,*}\omega\right)(x) = f(x)\,dx$$

for all $x \in \mathbb{R}$.

We must find this f!

Calculating, if $s \in \mathbb{R}$, $(x,s) \in \mathbb{R}_x$, and

$$\left(\left(I_{(2,0)}^{2\,*}\,\omega\right)(x)\right)(x,s)$$

$$=\left(I_{(2,0)}^{2\,*}\left(\omega\left(I^{2}_{(2,0)}(x)\right)\right)\right)(x,s)$$

$$=\left(I_{(2,0)}^{2\,*}\left(\omega(x,0)\right)\right)(x,s)$$

$$=\omega(x,0)\left(\left(I^{2}_{(2,0)}\right)_*(x,s)\right)$$

$$=\omega(x,0)\left(\underbrace{(x,0),\left(DI^{2}_{(2,0)}(x)\right)(x,s)}_{\in\,\mathbb{R}^2_{(x,0)}}\right)$$

Each equality on the previous page involves just tracing through the definitions.

It only remains to calculate

$$DI^2_{(2,0)}(x).$$

But since $I^2_{(2,0)}(x) = (x,0)$,

$$DI^2_{2,0}(x) = (1,0) \quad \forall x \in \mathbb{R}$$

and $(DI^2_{2,0}(x)) \cdot (x,s) = (x,0)$

Then

$$\left(\left(I^{2\,*}_{(2,0)}\omega\right)(x)\right)(x,s)$$

$$= \omega(x,0)\left((x,0),\left(DI^{2}_{(2,0)}(x)\right)(x,s)\right)$$

$$= \omega(x,0)\left((x,0),(x,0)\right)$$

$$= P(x,0)\,dx(x,0) + Q(x,0)\,dy(x,0)$$

$$= P(x,0) \quad \text{since } dy(x,0) = 0$$

Therefore, $f(x) = P(x,0)$ and

$$\int_{I^2_{(2,0)}} \omega$$

$$= \int_0^1 P(x,0)\,dx$$ ✓

The other 3 integrals are similar.

The End!